Indian Statistical Institute, Bangalore Centre
M.Math. (II Year) : 2010-2011
Semester I : Semestral Examination
Stochastic Processes

3.12.2010 Time: 3 hours. Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. State clearly the results you are using in your answers.

1. [3×5=15 marks] Prove or disprove:

(i) $P_n, n \geq 1, P, Q$ are probability measures on a complete separable metric space such that $P_n \perp Q$ for all n, and $P_n \Rightarrow P$ as $n \to \infty$. Then $P \perp Q$.

(ii) S is a complete separable metric space. Then $x \mapsto \delta_x$ is a continuous function from S into $\mathcal{P}(S)$.

(iii) P_y denotes the one dimensional Wiener measure starting at $y \in \mathbb{R}$. Then P_x and P_{2x} are equivalent measures (in the sense of mutual absolute continuity), where $x \neq 0$.

2. [10 marks] Let $h : S \to S'$ be a measurable function where S, S' are complete separable metric spaces. Let $P_n \Rightarrow P$ in S. Suppose there exists $D \in \mathcal{B}(S)$ such that $P(D) = 0$ and $D_h \subseteq D$ where D_h is the set of discontinuity points of h. Show that $P_n h^{-1} \Rightarrow P h^{-1}$.

3. [15 marks] Let $\{B(t) : t \geq 0\}$ be a real valued stochastic process. Let $0 < t_1 < t_2 < \cdots < t_k < \infty$ be fixed. Show that the following are equivalent.

(a) $B(t_1), B(t_2) - B(t_1), \cdots, B(t_k) - B(t_{k-1})$ are independent random variables having respectively $N(0, t_1), N(0, t_2 - t_1), \cdots, N(0, t_k - t_{k-1})$ distributions.

(b) $(B(t_1), B(t_2), \cdots, B(t_k))$ has the k-dimensional normal distribution $N(\bar{0}, \Sigma)$ where $\bar{0} = (0, 0, \cdots, 0), \Sigma_{ij} = \min\{t_i, t_j\}, 1 \leq i, j \leq k$.

4. [15 marks] For $t > 0, x \in \mathbb{R}^d$, Borel measurable function f on $\mathbb{R}^d$, define

$$(T_t)f(x) = E_x[f(X(t))],$$

whenever the right side makes sense; here E_x denotes expectation w.r.t. the d-dimensional Wiener measure starting at x, and $\{X(s) : s \geq 0\}$ is the canonical projection process. Show that T_t is a bounded self adjoint operator on $L^2(\mathbb{R}^d)$ with $\|T_t\| \leq 1$.

5. [15 marks] Let $T > 0$. Let $u(\cdot, \cdot) \in C_b([0,T] \times \mathbb{R}^d) \bigcap C_b^{1,2}((0,T) \times \mathbb{R}^d)$ solve the terminal value problem

$$\frac{\partial}{\partial s} u(s,x) + \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(s,x) + q u(s,x) = 0, \quad 0 < s < T, x \in \mathbb{R}^d,$$

$$\lim_{s \uparrow T} u(s,x) = f(x), \quad x \in \mathbb{R}^d,$$

where $q \in \mathbb{R}$ is a constant and $f(\cdot)$ is a bounded continuous function. Obtain an expression for $u(\cdot, \cdot)$ in terms of d-dimensional Wiener measure. (*Hint:* Consider $e^{qs} u(s,x)$.)

6. [10+10+15=35 marks] Let $D \subset \mathbb{R}^d$ be an open set. For $w \in C([0,\infty) : \mathbb{R}^d)$ set

$$\tau_D(w) = \inf\{t \geq 0 : w(t) \notin D\}.$$

(i) Let $w_n \to w$ in $C([0,\infty) : \mathbb{R}^d)$. Show that $\tau_D(w) \leq \liminf_{n \to \infty} \tau_D(w_n)$.

(ii) Show that $w \mapsto \tau_D(w)$ is not continuous in general.

(iii) Using the strong Markov property of Brownian motion show that the function

$$x \mapsto P_x(\tau_D < \infty), \quad x \in D$$

is harmonic in D, where P_y denotes the d-dimensional Wiener measure starting at y.